Singlet $VA\widetilde{V}$ correlator within the instanton vacuum model

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Abstract

The correlator of singlet axial-vector and vector currents in the external electromagnetic field is studied within the instanton liquid model of QCD vacuum. In the chiral limit we calculate the longitudinal $w_L^{(0)}$ and transversal $w_T^{(0)}$ with respect to axial-vector index invariant amplitudes at arbitrary momentum transfer q. It is demonstrated how the anomalous longitudinal part of the correlator is renormalized at low momenta due to the presence of the $U_A(1)$ anomaly.

1 Introduction

Consideration of the axial-vector A and vector V current-current correlator in the soft external electromagnetic field $\widetilde{V}$ is an important part of the calculations of the complicated light-by-light scattering amplitude related to the problem of accurate computation of higher order hadronic contributions to muon anomalous magnetic moment[1]. In this specific kinematics when one photon (V) with momentum $q_2 \equiv q$ is virtual and another one $(\widetilde{V})$ with momentum q_1 represents the external electromagnetic field and can be regarded as a real photon with the vanishingly small momentum q_1 only two invariant functions survive in linear in small q_1 approximation. It is convenient to parameterize the $VA\widetilde{V}$ correlator (Fig. 1) in terms of longitudinal w_L and transversal w_T with respect to axial current index Lorentz invariant amplitudes

$$\widetilde{T}_{\mu\nu\lambda}(q_1,q_2) = \frac{1}{4\pi^2}\left[w_T\left(q^2\right)\left(q_2^2 q_1^\rho \varepsilon_{\rho\mu\nu\lambda} - q_2^\nu q_1^\rho q_2^\sigma \varepsilon_{\rho\mu\sigma\lambda} + q_2^\lambda q_1^\rho q_2^\sigma \varepsilon_{\rho\mu\sigma\nu}\right) - w_L\left(q^2\right) q_2^\lambda q_1^\rho q_2^\sigma \varepsilon_{\rho\mu\sigma\nu}\right]. \tag{1}$$

Both Lorentz structures are transversal with respect to vector current, $q_2^\nu T_{\nu\lambda} = 0$. As for the axial current, the first structure is transversal with respect to q_2^λ while the second one is longitudinal and thus anomalous. The appearance of the longitudinal structure is the consequence of the Adler-Bell-Jackiw axial anomaly [3, 4].

For the nonsinglet axial current $A^{(3)}$ there are no perturbative [5] and nonperturbative [6] corrections to the axial anomaly and, as consequence, the invariant function $w_L^{(3)}$

[1] See, *e.g.,* [1, 2] and references therein.

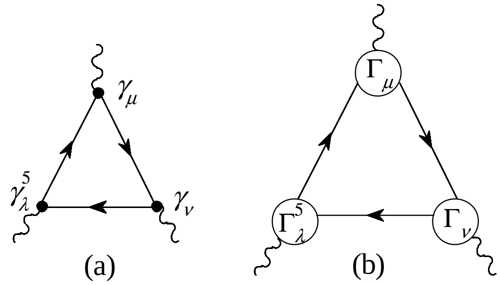

Figure 1: Diagrammatic representation of the triangle diagram in the local perturbative theory (a); and in the instanton model with dressed quark lines and full quark-current vertices (b).

remains intact when interaction with gluons is taken into account. It was shown in [7] that in the nonsinglet channel the transversal structure $w_T^{(3)}$ is also free from perturbative corrections. Nonperturbative nonrenormalization of the nonsinglet longitudinal part follows from the 't Hooft consistency condition [6], i.e. the exact quark-hadron duality realized as a correspondence between the infrared singularity of the quark triangle and the massless pion pole in terms of hadrons. However, for the singlet axial current $A^{(0)}$ due to the gluonic $U_A(1)$ anomaly there is no massless state even in the chiral limit. Instead, the massive η' meson appears. So, one expects nonperturbative renormalization of the singlet anomalous amplitude $w_L^{(0)}$ at momenta below η' mass.

In the previous work [8] we analyzed in the framework of the instanton liquid model [9] the nonperturbative properties of the nonsinglet triangle diagram in the kinematics specified above. We demonstrated how the anomalous structure $w_L^{(3)}$ is saturated within the instanton liquid model. We also calculated the transversal invariant function $w_T^{(3)}$ at arbitrary q and show that within the instanton model at large q^2 there are no power corrections to this structure. The nonperturbative corrections to $w_T^{(3)}$ at large q^2 have exponentially decreasing behavior related to the short distance properties of the instanton nonlocality in the QCD vacuum.

The present work is devoted to the study of the $U_A(1)$ anomaly effect on the singlet $VA\widetilde{V}$ correlator within the instanton liquid model. We calculate in the chiral limit the longitudinal $w_L^{(0)}$ and transversal $w_T^{(0)}$ invariant functions at arbitrary momentum transfer q and demonstrate how the singlet anomalous $w_L^{(0)}$ part of the correlator $w_L^{(0)}$ is renormalized at low momenta due to presence of the $U_A(1)$ anomaly.

2 The structure of $VA\widetilde{V}$ correlator in perturbative approach

The amplitude for the triangle diagram can be written as a correlator of the axial current j^5_λ and two vector currents j_ν and $\tilde{j}_\mu$ (Fig. 1)

$$\widetilde{T}_{\mu\nu\lambda} = -\int \mathrm{d}^4x\mathrm{d}^4y\, \mathrm{e}^{iqx-iky}\, \langle 0|\, T\{j_\nu(x)\, \tilde{j}_\mu(y)\, j^5_\lambda(0)\}|0\rangle\,, \tag{2}$$

where for light u and d quarks one has in the local theory

$$j_\mu = \bar{q}\,\gamma_\mu V q, \qquad j_\lambda^5 = \bar{q}\,\gamma_\lambda \gamma_5 A q\,,$$

the quark field q_f^i has color (i) and flavor (f) indices, $A^{(0)} = I$, $A^{(3)} = \tau_3$ are flavor matrix of the axial current, and $V = \widetilde{V} = \frac{1}{2}\left(\frac{1}{3} + \tau_3\right)$ are the charge matrices, with the tilted current being for the soft momentum photon vertex.

In the local perturbative theory the one-loop result (Fig. 1a) for the invariant functions w_T and w_L for space-like momenta q $(q^2 \geq 0)$ is

$$w_L^{1-\mathrm{loop}} = 2\, w_T^{1-\mathrm{loop}} = 2N_c \mathrm{Tr}\left(AV\widetilde{V}\right) \int_0^1 \frac{\mathrm{d}\alpha\, \alpha(1-\alpha)}{\alpha(1-\alpha)q^2 + m_f^2}\,, \tag{3}$$

where N_c is the color number, and for light quark masses one takes $m_f \equiv m_u \approx m_d$. In the chiral limit, $m_f = 0$, one gets the result

$$w_L\left(q^2\right) = 2w_T\left(q^2\right) = \frac{2N_c}{q^2}\mathrm{Tr}\left(AV\widetilde{V}\right). \tag{4}$$

3 The instanton effective quark model

To study nonperturbative effects in the triangle amplitude $\widetilde{T}_{\mu\nu\lambda}$ at low and intermediate momenta one can use the framework of the effective approach based on the representation of QCD vacuum as an ensemble of strong vacuum fluctuations of gluon field, instantons. Spontaneous breaking the chiral symmetry and dynamical generation of a momentum-dependent quark mass are naturally explained within the instanton liquid model. The instanton fluctuations characterize nonlocal properties of the QCD vacuum [10, 11, 12]. The interaction of light u, d quarks in the instanton vacuum can be described in terms of the effective 't Hooft four-quark action with nonlocal kernel induced by quark zero modes in the instanton field. The gauged version of the model [13, 14, 15] meets the symmetry properties with respect to external gauge fields, and corresponding vertices satisfy the Ward-Takahashi identities.

In the framework of this effective model the nonsinglet V and A current-current correlators, the vector Adler function, the pion transition form factor have been calculated for arbitrary current virtualities in [15, 16, 17]. In the same model the topological susceptibility of the QCD vacuum which is reduced to the singlet A current-current correlator has been considered in [15, 18].

The spin-flavor structure of the nonlocal chirally invariant interaction of soft quarks is given by the matrix products[2]

$$G\left(1 \otimes 1 + i\gamma_5\tau^a \otimes i\gamma_5\tau^a\right), \qquad G'\left(\tau^a \otimes \tau^a + i\gamma_5 \otimes i\gamma_5\right), \tag{5}$$

where G and G' are 4-quark couplings in iso-triplet and singlet channels. For the interaction in the form of 't Hooft determinant one has the relation $G' = -G$. In general due to repulsion in the singlet channel the relation $G' < G$ is required.

Within the gauged instanton model the dressed quark propagator in the chiral limit, $S(p)$, is defined as

$$S^{-1}(p) = i\widehat{p} - M(p^2), \tag{6}$$

[2] The explicit calculations below are performed in $SU_f(2)$ sector of the model.

with the momentum-dependent quark mass

$$M(p^2) = M_q f^2(p^2) \tag{7}$$

found as the solution of the gap equation

$$M(p^2) = 4G_P N_f N_c f^2(p^2) \int \frac{d^4k}{(2\pi)^4} f^2(k^2) \frac{M(k^2)}{D(k^2)}, \tag{8}$$

where we denote

$$D\left(k^2\right) = k^2 + M^2(k).$$

The constant $M_q \equiv M(0)$ in (7) is determined dynamically from Eq. (8) and the function $f(p)$ defines the nonlocal kernel of the four-quark interaction. Within the instanton model $f(p)$ describing the momentum distribution of quarks in the nonperturbative QCD vacuum is expressed through the quark zero mode function. It is implied in [8, 11, 12] that the quark zero mode in the instanton field is taken in the axial gauge when the gauge dependent dynamical quark mass is defined. In particular it means that $f(p)$ for large arguments decreases like some exponential in p^2. To make numerics simpler we shell use the Gaussian form

$$f(p) = \exp\left(-p^2/\Lambda^2\right), \tag{9}$$

where the parameter Λ characterizes the size of nonlocal fluctuations in the QCD vacuum and it is proportional to the inverse average size of an instanton.

The vector vertex following from the instanton model is [14, 15] (Fig. 2a)

$$\Gamma_\mu(k, k') = \gamma_\mu + (k + k')_\mu M^{(1)}(k, k'), \tag{10}$$

where $M^{(1)}(k, k')$ is the finite-difference derivative of the dynamical quark mass, q is the momentum corresponding to the current, and k (k') is the incoming (outgoing) momentum of the quark, $k' = k + q$. The finite-difference derivative of an arbitrary function F is defined as

$$F^{(1)}(k, k') = \frac{F(k') - F(k)}{k'^2 - k^2}. \tag{11}$$

The nonlocal part of the vertex (10) necessarily appears in order to fit the vector Ward-Takahashi identity.

Within the chiral quark model [14] based on the non-local structure of instanton vacuum [11] the full singlet axial-vector vertex including local and nonlocal pieces is given by [15]

$$\begin{aligned} \Gamma_\mu^{5(0)}(k, q, k' = k + q) = \gamma_\mu\gamma_5 + \gamma_5(k + k')_\mu M_q \frac{\left(f\left(k'\right) - f\left(k\right)\right)^2}{k'^2 - k^2} - \\ + \gamma_5 \frac{q_\mu}{q^2} 2M_q f\left(k'\right) f\left(k\right) \frac{G'}{G} \frac{1 - GJ_{PP}(q^2)}{1 - G'J_{PP}(q^2)}. \end{aligned} \tag{12}$$

where

$$J_{PP}(q^2) = \frac{8N_c}{M_q^2} \int \frac{d^4k}{(2\pi)^4} \frac{M_+M_-\left(k_+ \cdot k_- + M_+M_-\right)}{D_+D_-}. \tag{13}$$

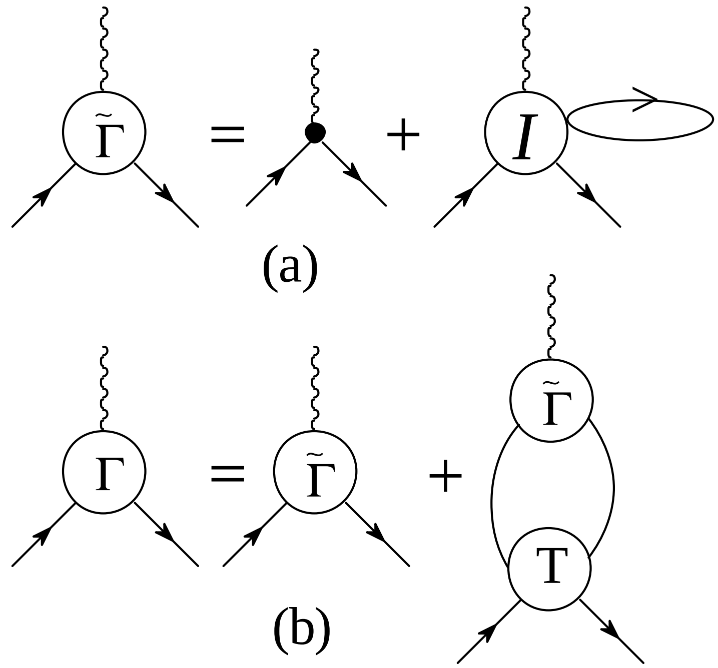

Figure 2: Diagrammatic representation of the bare (a) and full (b) quark-current vertices.

Here and below we use the notations

$$k_+ = k, \qquad k_- = k - q, \qquad k_\perp^2 = k_+ k_- - \frac{(k_+ q)\,(k_- q)}{q^2},$$
$$M_\pm = M(k_\pm^2), \qquad D_\pm = k_\pm^2 + M_\pm^2, \qquad f_\pm = f(k_\pm^2).$$

The vertex (12) takes into account the quark-antiquark rescattering in the singlet axial channel (Fig. 2b). For completeness we also present the vertex corresponding to the conserved iso-triplet axial-vector current which in the chiral limit is given by [14]

$$\Gamma_\mu^{5(3)}(k,k') = \tau_3 \left[\gamma_\mu \gamma_5 + 2\gamma_5 \frac{q_\mu}{q^2} M_q f(k) f(k') \right. \\ \left. + \gamma_5 (k+k')_\mu M_q \frac{\left(f(k') - f(k)\right)^2}{k'^2 - k^2} \right]. \tag{14}$$

The isotriplet axial-vector vertex has a massless pole, $q^2 = 0$, that follows from the spontaneous breaking of the chiral symmetry in the limit of massless u and d quarks. Evidently, this pole corresponds to the massless Goldstone pion.

The singlet current (12) does not contain massless pole due to presence of the $U_A(1)$ anomaly. Indeed, as $q^2 \to 0$ there is compensation between denominator and numerator in (12)

$$\frac{1 - G J_{PP}(q^2)}{-q^2} = G \frac{f_\pi^2}{M_q^2} \qquad \text{as} \quad q^2 \to 0, \tag{15}$$

where f_π is the pion weak decay constant. In cancellation of the massless pole the gap

equation is used. Instead, the singlet current develops a pole at the $\eta'-$ meson mass[3]

$$1-G'J_{PP}(q^2=-m_{\eta'}^2)=0, \tag{16}$$

thus solving the $U_A(1)$ problem. Let us also remind that in the instanton chiral quark model the connection between the gluon and effective quark degrees of freedom is fixed by the gap equation.

4 Singlet $VA\widetilde{V}$ correlator

In the effective instanton-like model the nondiagonal correlator of vector current and singlet axial-vector current in the external electromagnetic field ($VA\widetilde{V}$) is given by (Fig. 1b)

$$\widetilde{T}_{\mu\nu\lambda}(q_1,q_2)=-2N_c\mathrm{Tr}\left(AV\widetilde{V}\right)\int\frac{d^4k}{(2\pi)^4}\cdot \tag{17}$$
$$\cdot\mathrm{Tr}\left[\Gamma_\mu\left(k+q_1,k\right)S\left(k+q_1\right)\Gamma_\lambda^{5(0)}\left(k+q_1,k-q_2\right)S\left(k-q_2\right)\Gamma_\nu\left(k,k-q_2\right)S\left(k\right)\right],$$

where the quark propagator, the vector and the axial-vector vertices are defined by (6), (10) and (12), respectively. The structure of the vector vertex (10) guarantees that the amplitude is transversal with respect to vector indices

$$\widetilde{T}_{\mu\nu\lambda}(q_1,q_2)q_1^\mu=\widetilde{T}_{\mu\nu\lambda}(q_1,q_2)q_2^\nu=0$$

and the Lorentz structure of the amplitude is given by (1).

In [8] we found for the nonsinglet axial current $\Gamma_\lambda^{5(3)}$ the expressions for the longitudinal amplitude

$$w_L^{(3)}(q^2)=\frac{2N_c}{3}\frac{1}{q^2}, \tag{18}$$

and for the combination of invariant functions which shows up the nonperturbative dynamics

$$w_{LT}^{(3)}\left(q^2\right)\equiv w_L^{(3)}(q^2)-2w_T^{(3)}(q^2)=\frac{4N_c}{3q^2}\int\frac{d^4k}{\pi^2}\frac{\sqrt{M_-}}{D_+^2D_-}\left\{\sqrt{M_-}\left[M_+-\frac{2}{3}M_+'\left(k^2+2\frac{(kq)^2}{q^2}\right)\right]-\right.$$
$$\left.-\frac{4}{3}k_\perp^2\left[\sqrt{M_+}M^{(1)}(k_+,k_-)-2\left(kq\right)M_+'\sqrt{M}^{(1)}(k_+,k_-)\right]\right\}, \tag{19}$$

where prime means a derivative with respect to k^2: $M'(k^2)=dM(k^2)/dk^2$. The result (18) which is independent of the details of the nonlocal effective model is in agreement with the statement about absence of nonperturbative corrections to the nonsinglet longitudinal invariant function that follows from the 't Hooft duality arguments.

The calculations of the singlet $VA\widetilde{V}$ correlator results in the following modification of the nonsinglet amplitudes

$$w_L^{(0)}(q^2)=\frac{5}{3}w_L^{(3)}\left(q^2\right)+\Delta w^{(0)}\left(q^2\right), \tag{20}$$
$$w_{LT}^{(0)}(q^2)=\frac{5}{3}w_{LT}^{(3)}\left(q^2\right)+\Delta w^{(0)}\left(q^2\right), \tag{21}$$

[3]See previous footnote. Also we neglect the effect of the axial-pseudoscalar mixing with the longitudinal component of the flavor singlet f_1 meson.

where

$$\Delta w^{(0)}\left(q^2\right) = -\frac{5N_c}{9q^2}\frac{1-G'/G}{1-G'J_{PP}\left(q^2\right)}\int\frac{d^4k}{\pi^4}\frac{\sqrt{M_+M_-}}{D_+^2D_-}\left[M_+ - \frac{4}{3}M_+'k_\perp^2 - \right.$$
$$\left. -M^{(1)}(k_+,k_-)\left(\frac{4}{3}\frac{(kq)^2}{q^2}+\frac{2}{3}k^2-(kq)\right)\right]. \qquad (22)$$

In [8] it was shown that in the chiral limit the nonsinglet transversal amplitude gets only exponentially suppressed at large momenta corrections. The reason is that the asymptotics of the amplitude (19) is proportional to the vacuum nonlocality function $f(q)$ that is necessarily has exponentially decreasing asymptotics. The singlet amplitudes differ from the nonsinglet ones by the term (22) that also has exponentially suppressed large q^2 asymptotics. Thus, within the instanton model the singlet longitudinal and transversal parts have only exponentially suppressed at large q^2 corrections and all allowed by operator product expansion power corrections are canceled each other.

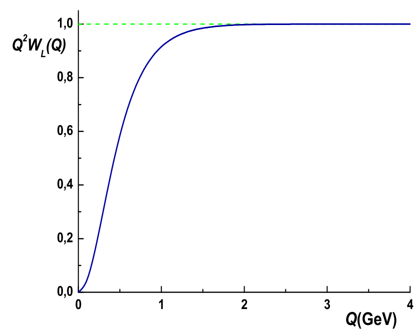

Figure 3: Normalized w_L invariant function in the singlet case (solid line) and nonsinglet case (dashed line).

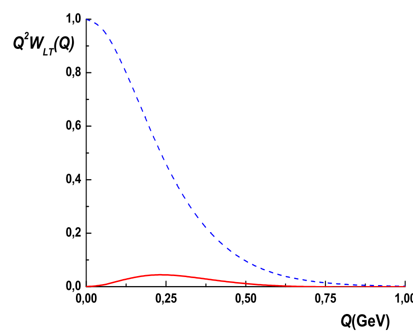

Figure 4: Normalized w_{LT} invariant function versus Q predicted by the instanton model in the singlet case (solid line) and isotriplet case (dashed line).

Fig. 3 illustrates how the singlet longitudinal amplitude $w_L^{(0)}$ is renormalized at low momenta by the presence of the $U_A(1)$ anomaly. The behavior of $w_{LT}^{(0)}(q^2)$ is presented in Fig. 4. In both figures the corresponding results for the nonsinglet case are also shown. The values of the model parameters used in calculations are fixed earlier in [8, 16] as

$$M_q = 0.24\ \text{GeV}, \qquad \Lambda_P = 1.11\ \text{GeV}, \quad G_P = 27.4\ \text{GeV}^{-2}. \qquad (23)$$

The coupling G' is fixed by fitting the meson spectrum. Approximately one has $G' \approx 0.1\,G$ [13]. We also find numerical values of the invariant amplitudes at zero virtuality

$$w_L^{(0)}(q^2=0) = 4.4\ \text{GeV}^{-2}, \qquad w_{LT}^{(0)}(q^2=0) = 0.6\ \text{GeV}^{-2}. \qquad (24)$$

Precise form and even sign of $w_{LT}^{(0)}(q^2)$ strongly depend on the ratio of couplings G'/G and has to be defined in the calculations with more realistic choice of model parameters.

5 Conclusions

In the framework of the instanton liquid model we have calculated for arbitrary momenta transfer the nondiagonal correlator of the singlet axial-vector and vector currents in the background of a soft vector field. For this specific kinematics we find that in the chiral limit the large momenta power corrections are absent for both longitudinal w_L as well transversal w_T invariant amplitudes. These amplitudes have very similar behavior and are corrected only by exponentially small terms which reflect the nonlocal structure of QCD vacuum.

Within the instanton model the renormalization of the singlet longitudinal $w_L^{(0)}$ amplitude occurring at low momenta due to the $U_A(1)$ anomaly is demonstrated explicitly. In the nonsinglet case the behavior of w_L and w_T at low momenta is very different due to the contribution of the massless pion state. At the same time in the singlet case there is no massless state and the deflection of w_L from $2w_T$ amplitudes is rather small.

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